Grundlagen der Rekonstruktion bei bildgebenden 3D-Schichtverfahren

Im Rahmen des 8. Fortbildungsseminars der Arbeitsgemeinschaften

Physik und Technik der DRG (APT) und der DGMP (K22) in Magdeburg

am 18. und 19. Juni 2004

Dipl. Math. Ulrich Storb

Fakultät für Naturwissenschaften

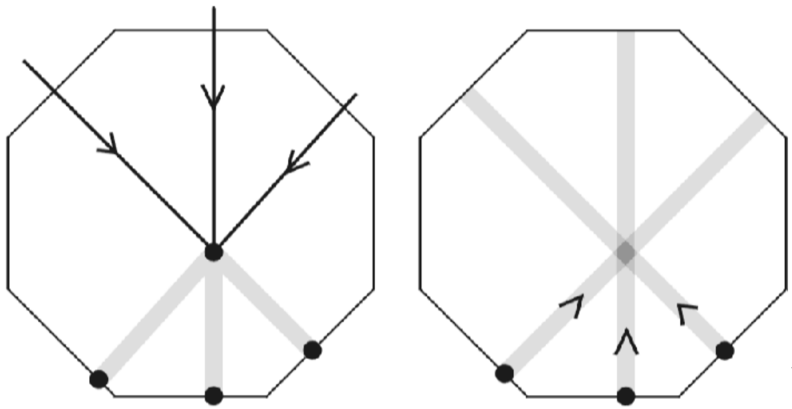
Abteilung Biophysik

Otto-von-Guericke-Universität Magdeburg

Abbildungen stammen teilweise aus:

1. *Gerthsen Physik*, 21. Auflage, D. Meschede, Springer-Verlag, 2002
2. *The Mathematics of Computerized Tomography*, F. Natterer, John Wiley & Sons and B G Teubner, 1986
3. *Quantitative Untersuchung dreidimensionaler Wellen in der Belousov-Zhabotinsky-Reaktion mittels optischer Tomographie*, D. Stock, Dissertation, Göttingen, 1996

Basics: Projections



Winfree et al., Chaos **6**:617, 1996

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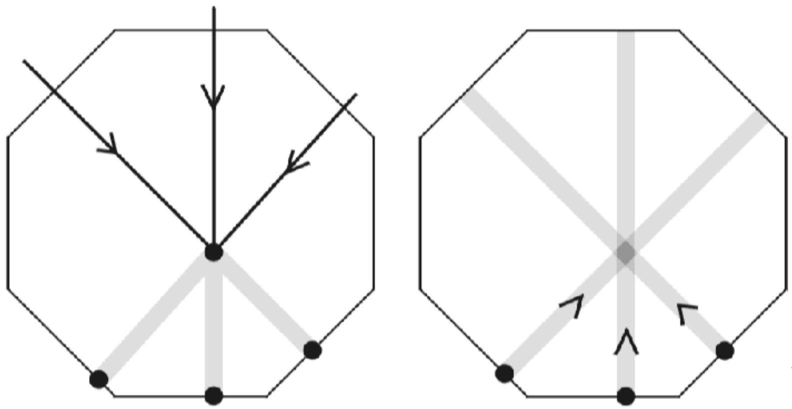

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- Location of a pointlike object can be recovered from projections along different lines
- Shape of the object is distorted anisotropic
- Distortion depends on
  - the number and directions of the lines
  - absorption properties of the object

Basics: Lambert-Beer's law

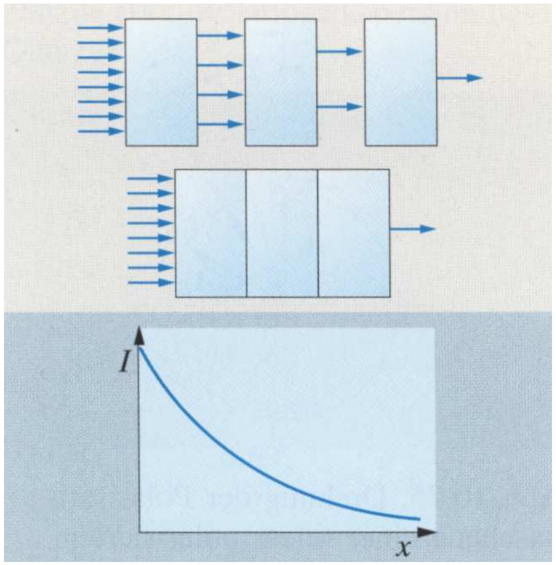

Fig. 10.77, Gerthsen Physik

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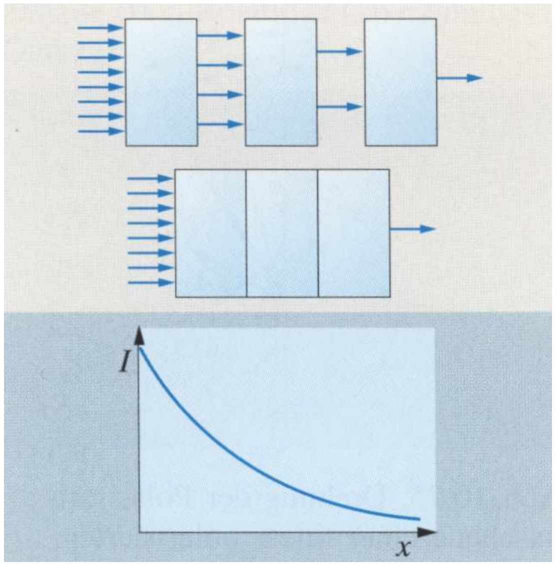


Fig. 10.77, Gerthsen Physik

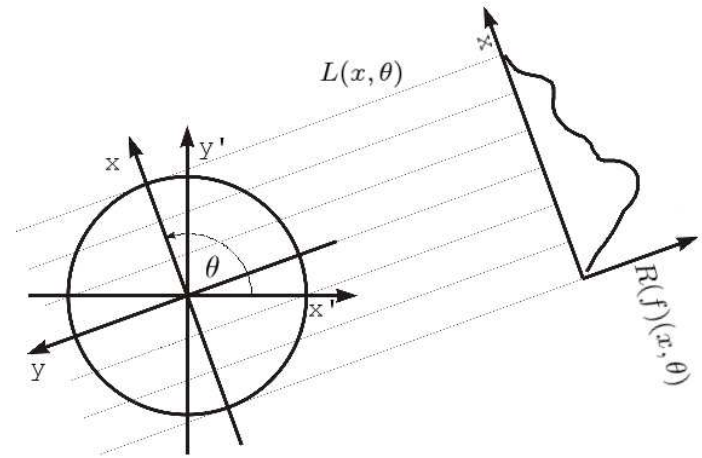
- Local absorption due to $\Delta I / \Delta x = -f I$
- Integral absorption does not depend on the partition of the set $\{x | f(x) = f_0\}$
- Initial and attenuated intensity are related by $I_1 / I_0 = exp\{- \int_L f(x) dx\}$

Basics: The Radon transform

$$\mathbf{R} : S(\mathbb{R}^2) \;\rightarrow\; S(\mathbb{G}^2)$$

$$\mathbf{R}(f) : (L) \;\mapsto\; \mathbf{R}(f)(L)$$

$$:= \int_{\{(x',y')\}\cap L\neq\emptyset} f(x',y')\,d\sigma$$



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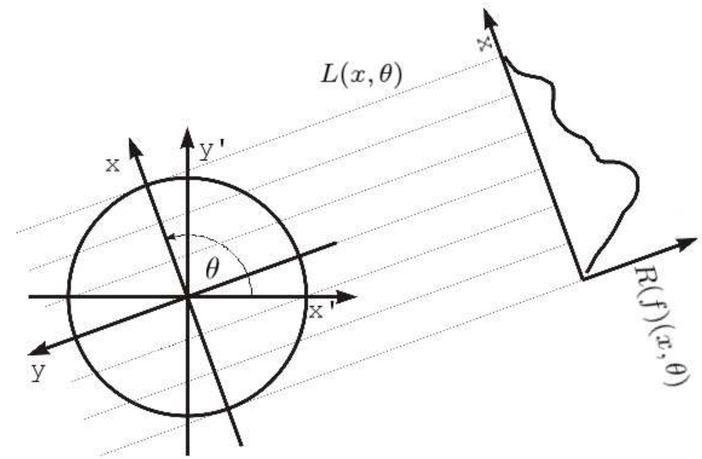
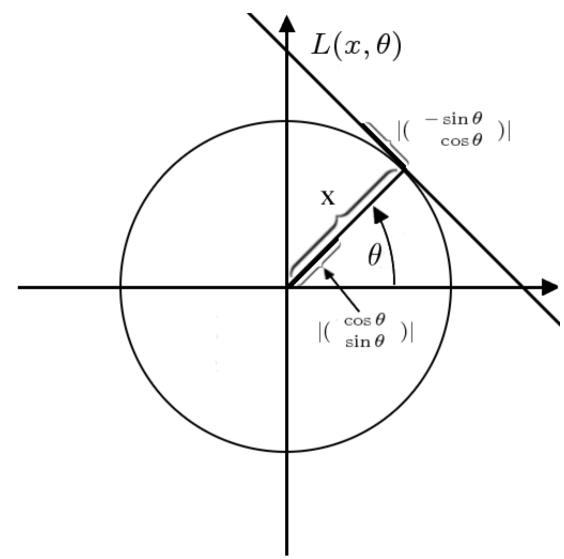


$\mathbb{G}^2 = \{$all lines in $\mathbb{R}^2\}$. $L\epsilon\mathbb{G}^2$ is parametrized by $(x,\theta)\epsilon\mathbb{R}\times S^1$, whereas the elements of $L(x,\theta)$ are parametrized by $y\epsilon\mathbb{R}$



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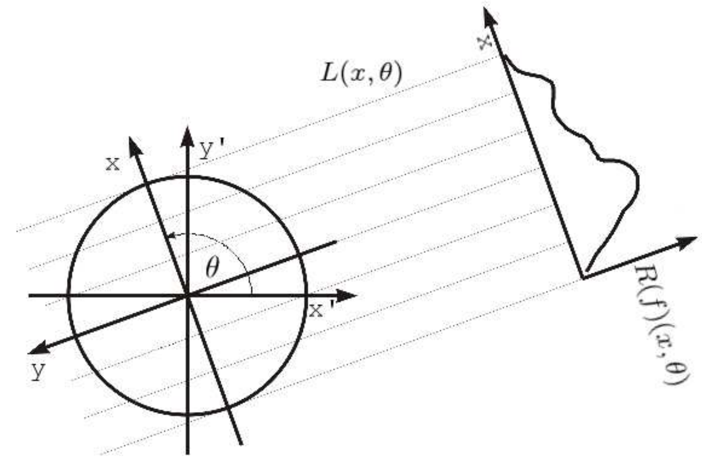

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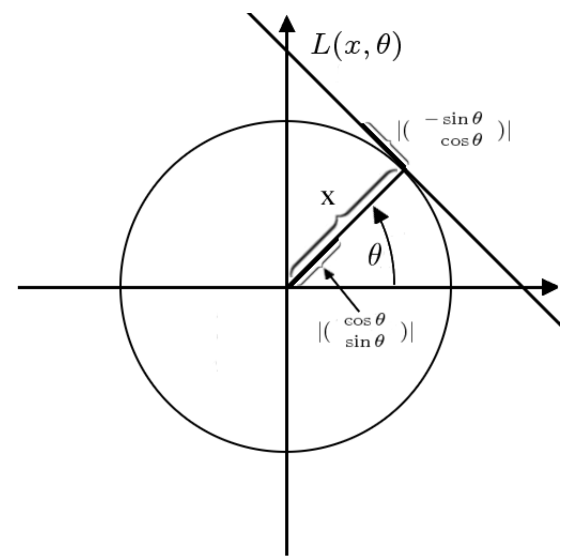
$$L(x,\theta) = \left\{ \; x\cdot\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + y\cdot\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix} \;\middle|\; y\in\mathbb{R} \;\right\}$$
$$= \left\{ \; \mathbf{D}(\theta)\cdot\begin{pmatrix}x\\ y\end{pmatrix} \;\middle|\; y\in\mathbb{R} \;\right\}$$



$\theta =$ angle between L and the vertical axis.

$x =$ distance from the origin in multiples of $|\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}|$.

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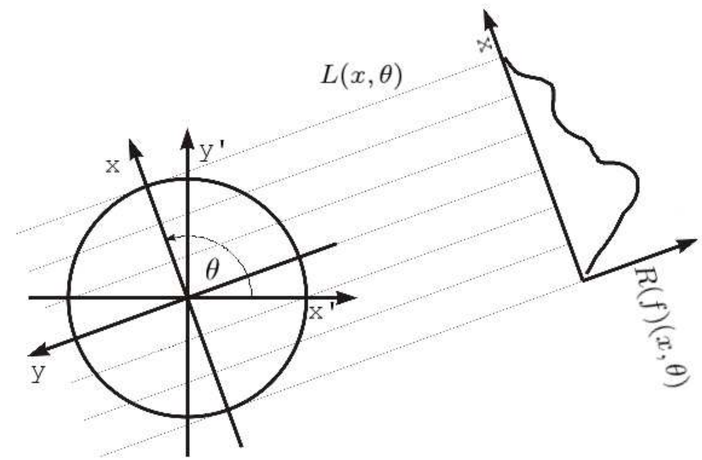

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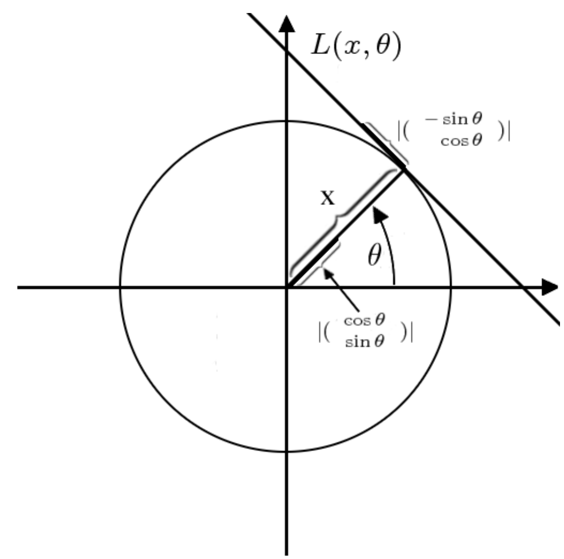

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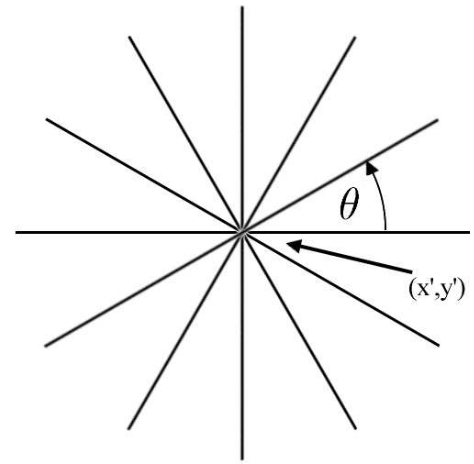
$x =$ distance from the origin in multiples of $|\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}|$.

$$d\sigma = |\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}|dy = dy$$
$$\Rightarrow \int_{\{(x',y')\}\cap L\neq\emptyset} f(x',y')d\sigma = \int_{\mathbb{R}} f\left(x\cdot\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} + y\cdot\begin{pmatrix}-\sin\theta\\ \cos\theta\end{pmatrix}\right)dy$$

Basics: The dual transform

$$\begin{aligned}\mathbf{P} : S(\mathbb{G}^2) &\rightarrow S(\mathbb{R}^2)\\ \mathbf{P}(g) : (x', y') &\mapsto \mathbf{P}(g)(x', y')\\ &:= \int_{\{(x',y')\}\cap L\neq\emptyset} g(L)d\sigma'\end{aligned}$$



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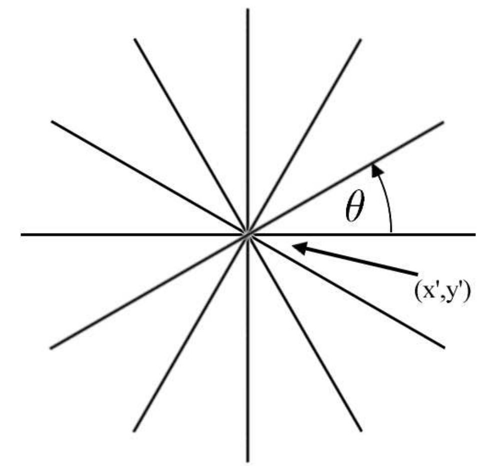
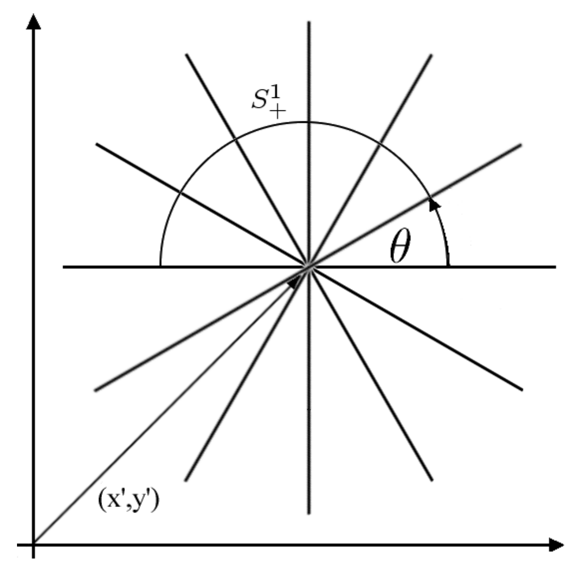


Lines passing through $\begin{pmatrix} x' \\ y' \end{pmatrix}$ are characterized by their intersection with the set $\begin{pmatrix} x' \\ y' \end{pmatrix} + S^1_+$.



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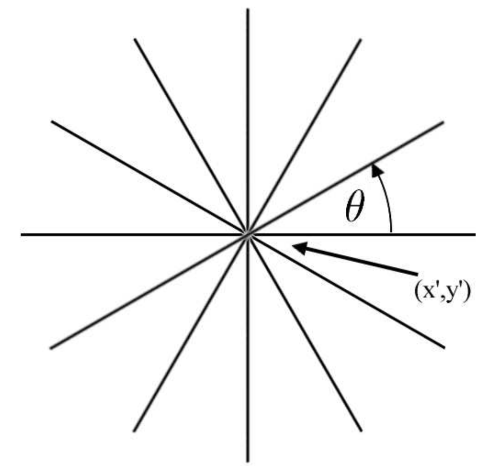

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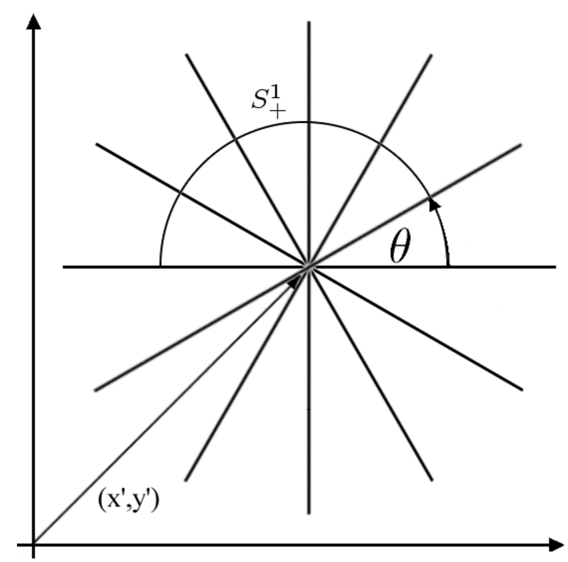
This set of intersection points is parametrized as:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} + S^1_+ = \{\begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \mid \theta \epsilon [0, \pi)\}$$



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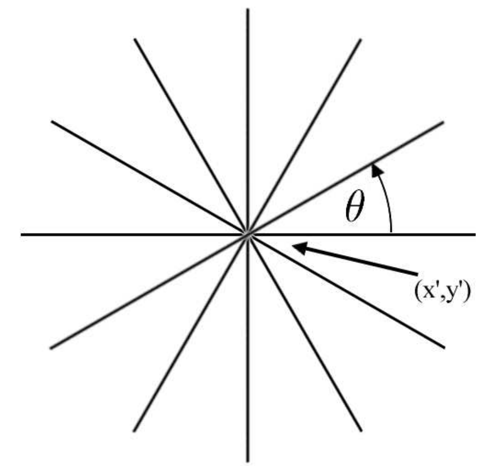

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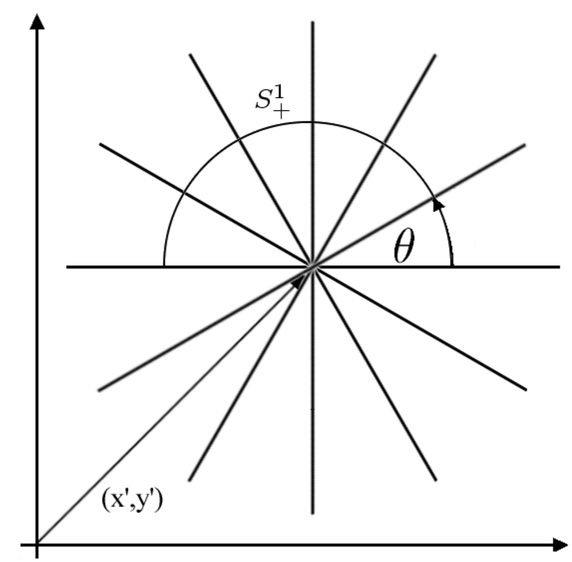
$$d\sigma' = |\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}| d\theta = d\theta$$

$$\Rightarrow \int_{\{(x',y')\}\cap L \neq \emptyset} g(L) d\sigma' = \int_0^\pi g(\begin{pmatrix} x' \\ y' \end{pmatrix} + \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}) d\theta$$

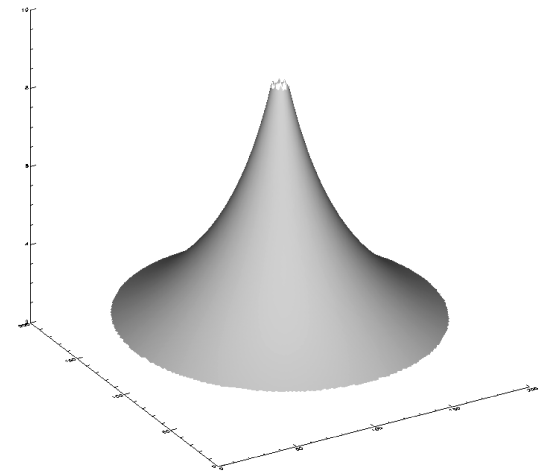


Basics: Inversion formulas

First step towards the calculation of the inverse:

Calculate $\mathbf{P} \circ \mathbf{R}(f)$

$$\begin{aligned} \mathbf{P} \circ \mathbf{R}(f)(x', y') &= \int_{\{(x',y')\} \cap L \neq \emptyset} \int_{\{(x,y)\} \cap L \neq \emptyset} f(x,y) d\sigma d\sigma' \\ &= \int_{\mathbb{R}^2} \frac{f(x,y)}{|(x',y') - (x,y)|} d^2(x,y) \\ &= \int_0^\pi \int_{\mathbb{R}} f(x' + y\cos\theta, y' + y\sin\theta) d\theta dy \end{aligned}$$



The action of $\mathbf{P} \circ \mathbf{R}$ on $f \epsilon S(\mathbb{R}^2)$ is the convolution with the

kernel $\frac{1}{|(x,y)|}$.

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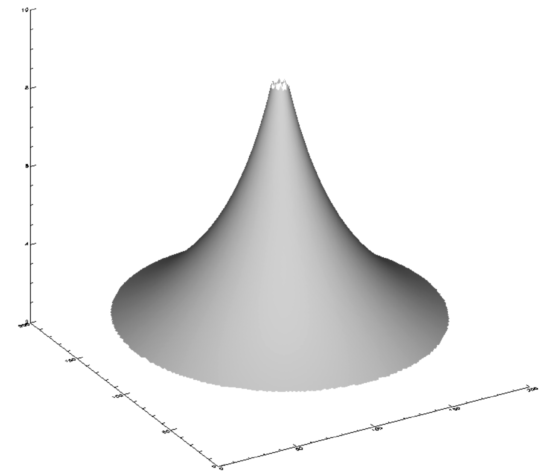

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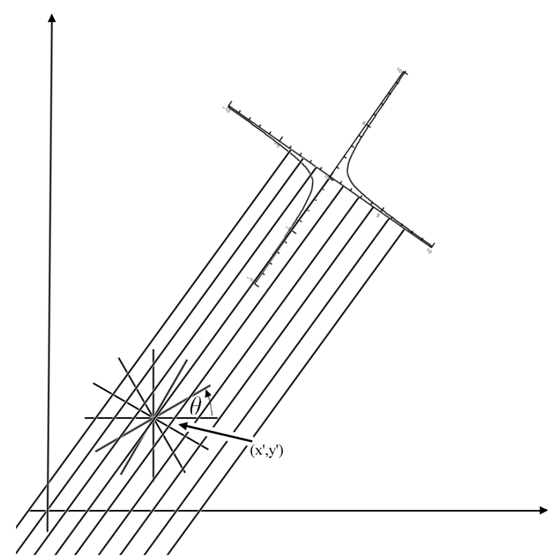
$$\begin{aligned} \mathbf{P} \circ \mathbf{R}(f)(x', y') &= \int_{\{(x',y')\}\cap L \neq \emptyset} \int_{\{(x,y)\}\cap L \neq \emptyset} f(x,y)\, d\sigma\, d\sigma' \\ &= \int_{\mathbb{R}^2} \frac{f(x,y)}{|(x',y')-(x,y)|} d^2(x,y) \\ &= \int_0^{\pi} \int_{\mathbb{R}} f(x' + y\cos\theta, y' + y\sin\theta)\, d\theta\, dy \end{aligned}$$

The action of $\mathbf{P} \circ \mathbf{R}$ on $f \epsilon S(\mathbb{R}^2)$ is the convolution with the kernel $\frac{1}{|(x,y)|}$. Taking, in between, for every $\{(x',y')\} \cap L \neq \emptyset$ the Cauchy principal value over all $L' || L$ yields:

$$\begin{aligned} &\mathbf{P} \circ \mathbf{H} \circ \mathbf{R}(f)(x', y') \\ &= \int_{\{(x',y')\}\cap L \neq \emptyset} \int_{L' || L} \frac{1}{dist(L,L')} \int_{\{(x,y)\}\cap L' \neq \emptyset} f(x,y)\, d\sigma\, d\sigma''\, d\sigma' \\ &= f(x,y) \end{aligned}$$

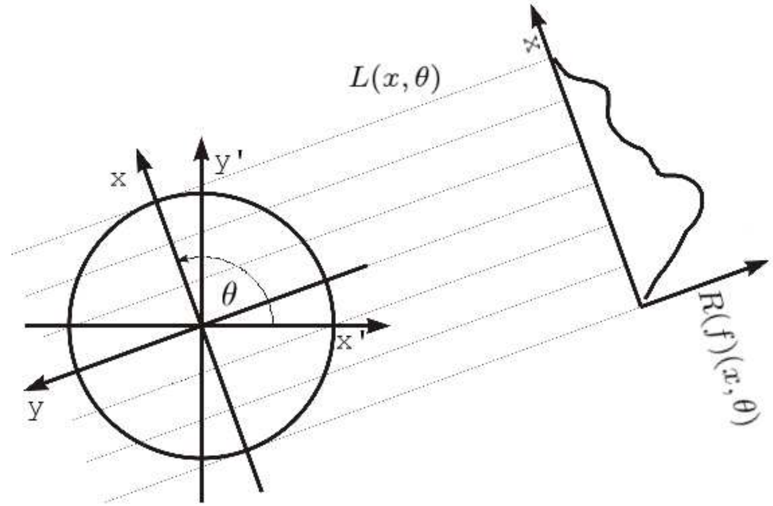


Basics: Inversion formulas

The Fourier transform $F(f)$ of the absorption distribution can be calculated on the basis of the Radon transform

$$
\begin{aligned}
& F(f)(s\cos\theta, s\sin\theta) \\
= \; & \frac{1}{2\pi}\int_{\mathbb{R}^2} f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)\cdot e^{-i\left\langle \begin{pmatrix} s\cos\theta \\ s\sin\theta \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle} dxdy \\
= \; & \frac{1}{2\pi}\int_{\mathbb{R}^2} f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right)\cdot e^{-i\left\langle s\cdot\mathbf{D}(\theta)\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle} dxdy \\
= \; & \frac{1}{2\pi}\int_{\mathbb{R}^2} f\left(\mathbf{D}(\theta)\begin{pmatrix} x \\ y \end{pmatrix}\right)\cdot e^{-isx} dxdy \\
= \; & \frac{1}{2\pi}\int_{\mathbb{R}} \left(\int_{\mathbb{R}} f\left(\mathbf{D}(\theta)\begin{pmatrix} x \\ y \end{pmatrix}\right) dy\right) e^{-isx} dx \\
= \; & \frac{1}{2\pi}\int_{\mathbb{R}} R(f)(x,\theta)\cdot e^{-isx} dx \, .
\end{aligned}
$$

Thereby s and θ take the role of the modulus and the direction of a wave vector.



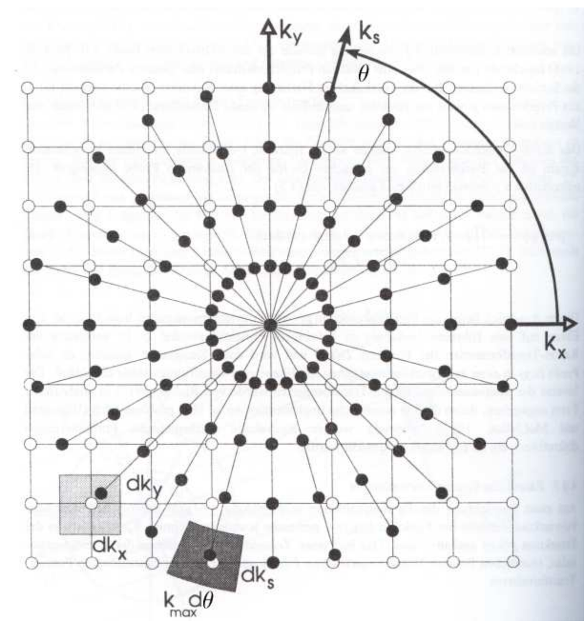


Fig. 4.7, D.Stock

Basics: Reconstruction results

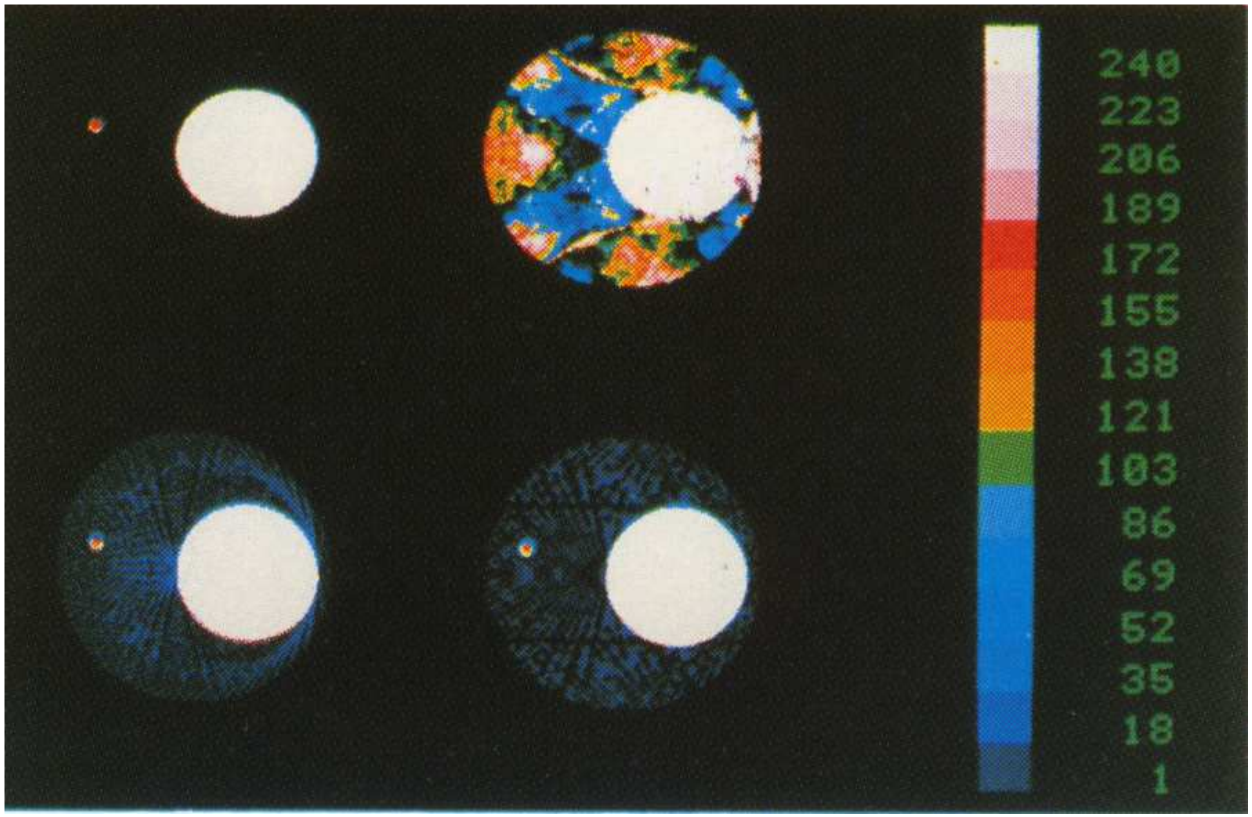


Fig. V.8, Natterer

Top left: Original image. **Bottom left:** Reconstruction by Filtered Backprojection.

Top right: Fourier Reconstruction. **Bottom right:** Refined Fourier Reconstruction.